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Waves induced by sources near the ocean surface

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The problem of wave generation by sources near the ocean surface is examined. The model is used to deduce that in the ocean, at low frequencies, both sound and surface gravity waves are possibly induced by the same sources. Sound radiation from particular sources is significantly influenced, at low frequencies, by surface waves which themselves are almost unmodified by fluid compressibility. The wave fields and energy levels are calculated analytically in a simple model of point-source excitation, which reveals considerable character. The theory is used to interpret measurements from the real ocean in the case where the source is a finite distribution of surface pressure fluctuations caused by winds. Good consistency is found in both structure and order of magnitude. Our main conclusions are that surface waves and wave-related sound are not necessarily cause and effect; a considerable proportion of the wave-associated underwater sound in the ocean is very likely generated by the same sources that produced the surface waves themselves.

1. Introduction

Waves on the ocean surface are known to be related to the structure of underwater sound. On the one hand, they can be sources of sound; surface waves may interact with each other to energize the sound field. This mechanism, involving nonlinear motions of the sea surface, has received much attention during the past two decades (e.g. Brekhovskikh 1966). On the other hand, sound and surface waves might possibly be inter-related because a source near the surface excites both kinds of waves. Should they originate from the same source it would be their common origin that gives their correlation; it is not one of cause and effect. This paper examines the detailed linear inter-connection between sound and surface waves.

Our problem is formulated in a strictly linear theory. The governing equation and boundary conditions are derived by taking into account both gravitational effects and fluid compressibility, and assuming the Brunt–Väisälä frequency to be zero so that internal gravity waves are precluded.

We start with an idealized problem in which a harmonic point source is assumed beneath the ocean surface. The response of the ocean to such an excitation is sought analytically. The results show that both sound and surface waves are induced. In this particular model, compressive waves and surface waves naturally coexist. Surface waves, propagating horizontally and decaying exponentially with depth, are unaffected by the compressibility of water. The dispersion relation is identical to that in classic water–wave theory (Lamb 1932) and surface waves travel at exactly the same speed as they did in incompressible flow. As for sound waves, the situation is quite different. At high frequencies, sound and surface waves are essentially decoupled; ocean-sound problems can then be solved separately from hydrodynamic problems (cf. Lighthill 1978). However, sound radiation from a near-surface source can be significantly altered, at low frequencies, by gravity waves on the surface. Without gravity, the constant-pressure free surface would have a reflection coefficient of -1; sound from the primary source is augmented by that from its negative image. However, gravity changes all that. Sound waves from the real source are reflected by coherent surface waves. Therefore, although still remaining pressure-release, the ocean surface reflects sound waves as if it had a reflective property that is a function of both frequency and observation position.

We also examine the energetics of this statically stratified model. It can be converted into a uniform mean-density model through an analogous velocity potential. The mean energy flux can then be calculated, in terms of this analogous velocity potential, in the same way as in gravity-free situations. We will show that the response of the ocean to the source is in two parts, one a wavelike motion that transports energy and the other a vertical background oscillation. The latter makes no contribution to the mean wave power output. Energy radiated from the point source to sound and surface waves is calculated explicitly. We were intrigued to find that the total acoustic power from the point source may be vanishingly small when the source is positioned at a particular depth in water. This is a striking feature that would never occur in gravity-free problems, and is entirely due to the change of reflective properties of the water surface, which provides an extra 'image' field that interferes destructively with the original one. The power calculations also show that deep water sources are more efficient in generating sound than producing surface waves, while sources in the proximity of the surface radiate more energy to surface waves than to sound.

Pressure fluctuations caused by winds moving across the ocean surface play a dominant role in the production of surface waves (Phillips 1977). We think that their direct effect on sound generation has not been adequately addressed, though Isakvich & Kur'yanov (1970) have recognized the possibility that the wind-associated pressure fluctuations are responsible for the underwater noise in the frequency range 10–50 Hz. We will examine this aspect in detail here and show that surface pressures can also generate an appreciable component of the low-frequency ocean sound below 10 Hz. We calculate the sound and surface wave power radiated from a finite distribution of sources prescribed on the surface. Our model of the wind-induced pressure is modelled from experiments. It is found that the calculated power ratio of the two kinds of waves agrees well with measurements in both the structure of frequency dependence and the order of magnitude. This then leads us to advocate the view that underwater sound and surface waves are probably linearly interconnected. By this, we mean that the underwater sound is a linear functional of the surface wave field. If the source were a surface pressure field, then a doubling of the surface pressure (keeping the space and time characteristics constant) would double the strength of both the underwater sound and the surface waves. This is contrary to the alternative view that surface waves themselves 'generate' the sound and do so by a nonlinear mechanism (Brekhovskikh 1966). A doubling of the surface wave activity in that event implies a quadrupling of the acoustic field. The experimental observation, that sound and surface waves increase by the same amount due to the increase of wind, indicates that a considerable proportion of the wave-related underwater noise is very likely generated by the same sources that produced the surface waves.

2. Waves induced by a harmonic point source

To bring out explicitly the linear connection between sound and surface waves, we consider here an idealized model of a statically stratified ocean under an infinite free surface, beneath which a harmonic point source is positioned at $(0, 0, -h)$, h being positive. Axes are selected so that the undisturbed water surface lies in $y_3 = 0$; water occupies $y_3 < 0$, and pressure above the water surface is assumed constant. The ocean water is supposed to be inviscid and originally at rest. The problem may then be formulated in terms of the velocity potential $\psi(\boldsymbol{y}, \tau)$. Furthermore, we suppose the entropy per unit mass to be uniform, which allows the use of $p' = c_w^2 \rho'$ as the definition of the constant sound speed c_w, p' and ρ' respectively representing pressure and density fluctuations in water.

The equations of motion are linearized and a source is assumed to have a constant strength Q and a time dependence $\exp(-i\omega\tau)$ with ω positive, this factor being suppressed throughout. Then the combination of mass and momentum conservation equations gives

$$\nabla^2\psi + \frac{\omega^2}{c_w^2}\psi - \frac{g}{c_w^2}\frac{\partial\psi}{\partial y_3} = Q\delta(y_\alpha, y_3 + h) \quad \text{in } y_3 < 0, \tag{2.1}$$

where g is the value of the gravitational acceleration. This equation takes simple account of gravity, in that internal gravity waves are precluded in this model for which the Brunt–Väisälä frequency vanishes.

The linearized boundary conditions at $y_3 = 0$ can be derived from the assumption that pressure on the surface is constant and the fact that particles in the free surface always remain in it. These conditions imply that

$$-\omega^2\psi + g\frac{\partial\psi}{\partial y_3} = 0 \quad \text{at } y_3 = 0. \tag{2.2}$$

At large depth, we impose a radiation condition to constrain disturbances to be either bounded or outgoing. Then we make the change of variables defined by

$$\psi(\boldsymbol{y}) = \left[\frac{\bar{\rho}}{\rho_w}\right]^{-\frac{1}{2}} \phi(\boldsymbol{y}), \tag{2.3}$$

where $\bar{\rho}(y_3) = \rho_w \exp(-g y_3/c_w^2)$ is the mean density, ρ_w being its value at the undisturbed surface. The physical meaning of this transformation will be discussed later, but it reduces (2.1) and (2.2) to

$$\nabla^2\phi + \lambda^2\phi = \delta(y_\alpha, y_3 + h) \quad \text{in } y_3 < 0, \tag{2.4}$$

and

$$(2\omega_0^2 - \omega^2)\phi + g\frac{\partial\phi}{\partial y_3} = 0 \quad \text{at } y_3 = 0, \tag{2.5}$$

where $\lambda^2 = (\omega^2 - \omega_0^2)/c_w^2$, and $\omega_0 = g/2c_w$ is the acoustical cutoff frequency for the ocean. The source strength Q has been chosen so that the coefficient of the Dirac delta function here is unity.

Take Fourier transforms in the horizontal coordinates y_α, so that

$$\phi(y_\alpha, y_3) = \frac{1}{(2\pi)^2}\int_\infty \hat{\phi}(k_\alpha, y_3)\, e^{-ik_\alpha y_\alpha}\, d^2k_\alpha,$$

and

$$\hat{\phi}(k_\alpha, y_3) = \int_\infty \phi(y_\alpha, y_3)\, e^{ik_\alpha y_\alpha}\, d^2y_\alpha.$$

Then (2.4) and (2.5) imply that

$$\frac{\partial^2\hat{\phi}}{\partial y_3^2}+\gamma^2\hat{\phi} = \delta(y_3+h) \quad \text{in } y_3 < 0, \tag{2.6}$$

and

$$(2\omega_0^2-\omega^2)\,\hat{\phi}+g\frac{\partial\hat{\phi}}{\partial y_3} = 0 \quad \text{at } y_3 = 0 \tag{2.7}$$

where $\gamma^2 = \lambda^2-k_\alpha^2$. In order to satisfy conditions at $y_3 \to -\infty$, we must take as solution of (2.6)

$$\hat{\phi} = A\,\mathrm{e}^{-\mathrm{i}\gamma y_3}+\frac{1}{2\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}\gamma|y_3+h|},$$

where A is a constant and γ is chosen so that when real it is positive, and when γ is purely imaginary $\mathrm{Im}\,(\gamma)$ is positive. From (2.7) we deduce

$$A = \frac{-1}{2\mathrm{i}\gamma}\,\frac{2\omega_0^2-\omega^2+\mathrm{i}g\gamma}{2\omega_0^2-\omega^2-\mathrm{i}g\gamma}\,\mathrm{e}^{\mathrm{i}\gamma h}.$$

The field $\phi(\boldsymbol{y})$ may now be calculated by inverse Fourier transforms, which yield

$$\phi = \frac{1}{(2\pi)^2}\int_\infty \frac{1}{2\mathrm{i}\gamma}\,\mathrm{e}^{\mathrm{i}[\gamma|y_3+h|-k_\alpha y_\alpha]}\,\mathrm{d}^2k_\alpha+\frac{1}{(2\pi)^2}\int_\infty \frac{1}{2\mathrm{i}\gamma}\,\mathrm{e}^{\mathrm{i}[\gamma|y_3-h|-k_\alpha y_\alpha]}\,\mathrm{d}^2k_\alpha$$
$$+\frac{1}{(2\pi)^2}\int_\infty \frac{\omega^2-2\omega_0^2}{\mathrm{i}\gamma(2\omega_0^2-\omega^2-\mathrm{i}g\gamma)}\,\mathrm{e}^{-\mathrm{i}[\gamma(y_3-h)+k_\alpha y_\alpha]}\,\mathrm{d}^2k_\alpha. \tag{2.8}$$

We denote these three terms respectively by ϕ_1, ϕ_2 and ϕ_3; ϕ_1 is the direct radiation from the source while ϕ_2 and ϕ_3 represent the field due to boundary reflection and gravity effects. The first two can be evaluated in a straightforward manner with the results

$$\phi_1 = \frac{-1}{4\pi R_1}\,\mathrm{e}^{\mathrm{i}\lambda R_1}, \quad \phi_2 = \frac{-1}{4\pi R_2}\,\mathrm{e}^{\mathrm{i}\lambda R_2}, \tag{2.9}$$

where $R_1 = (y_\alpha^2+(y_3+h)^2)^{\frac{1}{2}}$ and $R_2 = (y_\alpha^2+(y_3-h)^2)^{\frac{1}{2}}$. To find the third term, we convert the integral into polar coordinates according to $k_1 = k\cos\alpha$ and $k_2 = k\sin\alpha$, so that

$$\phi_3 = \frac{\omega^2-2\omega_0^2}{(2\pi)^2}\int_0^\infty\int_0^{2\pi} \frac{1}{\mathrm{i}\gamma(2\omega_0^2-\omega^2-\mathrm{i}g\gamma)}\,\mathrm{e}^{-\mathrm{i}[\gamma(y_3-h)+kr\cos(\alpha-\sigma)]}\,k\,\mathrm{d}k\,\mathrm{d}\alpha,$$

where $r = |y_\alpha|$, $\sigma = \mathrm{arctg}\,(y_2/y_1)$ and $\gamma^2 = \lambda^2-k^2$.

The α-integral can be evaluated explicitly as a Bessel function of zero order, $J_0(z)$, and since we are concerned here with the propagating waves, it is desirable to express the result in terms of the Hankel function. So we have

$$\phi_3 = \frac{2\omega_0^2-\omega^2}{4\pi\mathrm{i}}\int_k \frac{1}{\gamma(2\omega_0^2-\omega^2-\mathrm{i}g\gamma)}\,\mathrm{e}^{-\mathrm{i}\gamma(y_3-h)}\,H_0^{(1)}(kr)\,k\,\mathrm{d}k, \tag{2.10}$$

where $H_0^{(1)}(z)$ is the first-kind Hankel function of zeroth order. The integration path of (2.10) is the real k-axis indented above all singularities on the negative half axis and below all on the positive half.

For convenience, we suppose for the time being that $\omega > \omega_0 = g/2c_\mathrm{w}$. The integrand of (2.10) then has real singularities only, two poles $k = \pm\omega^2/g$ and three branch points $k = \pm\lambda$ and $k = 0$. The branch cuts from $k = \pm\lambda$ can be determined by the specification of γ, which guarantees that contributions to ϕ_3 for large negative

y_3 are bounded (for γ pure imaginary) or represent outgoing waves (for γ real). It turns out that the cut from $+\lambda$ must go to infinity above the real axis and that from $-\lambda$ to infinity below the real axis. The branch cut associated with the Hankel function can be drawn from $k = 0$ to infinity along the negative imaginary axis.

All these considerations about the integration path and branch cuts are made so that the radiation condition is satisfied. This can be understood by Lighthill's procedure (1978) that introduces an imaginary part into the frequency; we consider ω to be complex, $\omega + \mathrm{i}\epsilon$ say, ϵ being small and positive and later allowed to vanish. So, in Lighthill's scheme, the harmonic source strength increases exponentially with time like $\exp(\epsilon\tau)$, from zero at $\tau = -\infty$ to its present level. Since the linear response to the source must share the source time dependence, the amplitude of the response will also grow exponentially in step with the source strength. Hence the free waves which we wish to exclude are easily recognized on account of their negligible amplitude compared with the ever-growing source-generated fields. Accordingly we seek only the solutions proportional to $\exp(\epsilon\tau)$.

Mathematically the substitution of $\omega + \mathrm{i}\epsilon$ for ω moves singularities, both poles and branch points, on the negative real k-axis into the third quadrant, which is equivalent to the indenting of the integration path above the singularities. Similarly it moves singularities on the positive real axis into the first quadrant, an equivalence to indentation of the integration path below the singularities. Since the branch cuts cannot cross the integration path on which the integrand concerned is required to be analytic, the only way of drawing the cuts is then to let the one from $+\lambda$ go to infinity above the real axis and that from $-\lambda$ go to infinity below the axis, as illustrated in figure 1 where the loci of poles and the indented integration path are also schematically shown.

Now, we deform the integration path onto a straight line just above the whole real axis (except at the branch point $k = +\lambda$ where the new path is still indented into a semi-circle). In doing so, the pole $k = \omega^2/g$ is crossed. Hence ϕ_3 becomes the sum of the residue contribution at the pole and the integral along the new path, which we respectively denote by ϕ_s and ϕ_a,

$$\phi_\mathrm{s} = \frac{\mathrm{i}(\omega^2 - 2\omega_0^2)}{2g} \exp\left[\frac{\omega^2 - 2\omega_0^2}{g}(y_3 - h)\right] H_0^{(1)}\left(\frac{\omega^2}{g} r\right), \tag{2.11}$$

and ϕ_a is still given by (2.10) with the newly defined integration path. It can be calculated asymptotically by change of variables, the details of which are given in Appendix A. It is found there that

$$\phi_\mathrm{a} \sim \frac{2\omega_0^2 - \omega^2}{2\pi R_2} \mathrm{e}^{\mathrm{i}\lambda R_2} \sum_{n=0}^{\infty} \frac{\partial^n E(0)}{\partial s^n} \frac{1}{(\lambda R_2)^n} \quad (R_2 \to \infty), \tag{2.12}$$

where $\partial^n E(0)/\partial s^n$ is the nth-order derivative of $E(s)$ at $s = 0$ with $E(s)$ defined by

$$E(s) = [s^2 - 2g\lambda s[\mathrm{i}g\lambda + (2\omega_0^2 - \omega^2)\cos\theta_2] + [2\omega_0^2 - \omega^2 + \mathrm{i}g\lambda\cos\theta_2]^2]^{-\frac{1}{2}} \tag{2.13}$$

where $$\theta_2 = \operatorname{arctg}\left[\frac{y_3 - h}{r}\right].$$

It should be pointed out that we have assumed $\omega > \omega_0$ in the foregoing calculations. However it is easy to show that this restriction may be removed, because the solution for $\omega < \omega_0$ has exactly the same expressions as those obtained above. Therefore, the

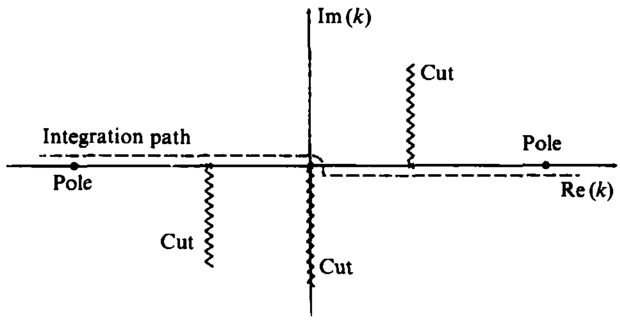

FIGURE 1. The loci of poles, branch cuts and the indented integration path in the complex k-plane.

complete solution of the problem (2.4) and (2.5) is given by (2.9), (2.11) and (2.12), asymptotically as

$$\phi(y) = \frac{-1}{4\pi R_1}\,\mathrm{e}^{\mathrm{i}\lambda R_1} + \frac{1}{4\pi R_2}\,\frac{2\omega_0^2-\omega^2-\mathrm{i}g\lambda\cos\theta_2}{2\omega_0^2-\omega^2+\mathrm{i}g\lambda\cos\theta_2}\,\mathrm{e}^{\mathrm{i}\lambda R_2} + \frac{2\omega_0^2-\omega^2}{2\pi R_2}\,\mathrm{e}^{\mathrm{i}\lambda R_2}\sum_{n=1}^{\infty}\frac{\partial^n E(0)}{\partial s^n}\,\frac{1}{(\lambda R_2)^n}$$
$$+\frac{\mathrm{i}(\omega^2-2\omega_0^2)}{2g}\exp\left[\frac{\omega^2-2\omega_0^2}{g}(y_3-h)\right]H_0^{(1)}\left(\frac{\omega^2}{g}r\right), \qquad (2.14)$$

where the second term is the combination of ϕ_2 and the first term ($n = 0$) in the series of (2.12). We rewrite them in this way because the combined term gives the far field ($\lambda R_2 \rightarrow \infty$) approximation for the reflected acoustic waves.

Both sound and surface waves are excited by the point source; the first three terms describe sound waves and the last represents gravity surface waves which propagate horizontally. Due to gravitational effects, sound waves are cut off below the frequency ω_0, where the wavenumber λ becomes purely imaginary , and they are dispersive; their phase speed varies very rapidly with frequency from infinity at $\omega = \omega_0$ to the constant value c_w as ω increases. But these interactive features are confined to the extremely low-frequency region near $\omega_0 = g/2c_w \approx 0.003$ rad/s so that we may ignore them for frequencies of practical interest.

Surface waves are almost unaffected by fluid compressibility. The dispersion relation of surface waves, determined by letting the denominator of the integrand of (2.10) vanish, is identical to that of classical water-wave theory; surface waves propagate at exactly the same speed as they did in incompressible flow. This is expected because the response field near the surface actually satisfies $\nabla^2\psi = 0$, the governing equation for incompressible flow, from (2.1) to (2.2).

However, though surface waves are unaware of fluid compressibility, their presence on the sea surface can greatly influence the compressive waves; they can drastically augment sound radiation from the source. This is readily seen from our solution. In the far field, where $\lambda R_2 \gg 1$, the sound field may be approximated by the first two terms of (2.14); the first is the field radiated directly from the source and the second the reflection from the ocean surface, the reflection coefficient being

$$C_r = -\frac{2\omega_0^2-\omega^2-\mathrm{i}g\lambda\cos\theta_2}{2\omega_0^2-\omega^2+\mathrm{i}g\lambda\cos\theta_2}. \qquad (2.15)$$

A pressure-release surface has a reflection coefficient of -1, but here, though the actual ocean surface still remains pressure free, its reflective property is a function of frequency as well as observation position. This is because the coherent gravity waves on the surface interfere with the otherwise specular reflection. Of course, $C_r = -1$ in the limiting case of vanishing gravity, and when the observation point approaches the surface, $\cos\theta_2 \approx 0$; sound waves cannot travel parallel to a linearly disturbed, passive surface without suffering severe cancellation. C_r also reduces to -1 at high frequencies where sound and gravity waves are uncoupled. The fact that a pressure-release surface can be regarded as having a locally reacting surface impedance when gravitational effects are considered has also been recently recognized by Finnveden (1987).

3. Energetics

The energy relation in our model can be derived in the same way as in gravity-free situations (Dowling & Ffowcs Williams 1983). The linearized governing equation for our problem is

$$\nabla^2\psi - \frac{1}{c_w^2}\frac{\partial^2\psi}{\partial\tau^2} - \frac{g}{c_w^2}\frac{\partial\psi}{\partial y_3} = 0. \tag{3.1}$$

When this is multiplied by $\bar{\rho}(y_3)\partial\psi/\partial\tau$, and the terms re-arranged, the equation describing the energy conservation is obtained as

$$\frac{\partial}{\partial\tau}(e_p + e_k) + \nabla\cdot\boldsymbol{I} = 0,$$

where

$$e_p = \frac{p'^2}{(2c_w^2\bar{\rho})}, \quad e_k = \frac{(\nabla\psi)^2}{(2\bar{\rho})},$$

and

$$\boldsymbol{I} = p'\nabla\psi = -\bar{\rho}\frac{\partial\psi}{\partial\tau}\nabla\psi. \tag{3.2}$$

It is apparent that for a steady wave field in which $\boldsymbol{I}$ is solenoidal the velocity potential ψ decays exponentially with depth because of the exponential increase of $\bar{\rho}(y_3)$. This property also manifests itself in the governing equation (3.1) by the first-order derivative term $-g/c_w^2(\partial\psi/\partial y_3)$ which results in an exponentially decaying solution.

In the previous section, we have introduced a new variable ϕ, having the same dimension as velocity potential. The physical meaning of it becomes clear once its definition (2.3) is substituted into (3.2), which yields

$$\boldsymbol{I} = -\rho_w\frac{\partial\phi}{\partial\tau}\nabla\phi - \rho_w\frac{g}{2c_w^2}\phi\frac{\partial\phi}{\partial\tau}\hat{\boldsymbol{y}}_3, \tag{3.3}$$

where $\hat{\boldsymbol{y}}_3$ is a unit vector in the vertically upward direction. Since ϕ and $\partial\phi/\partial\tau$ are always $\frac{1}{2}\pi$ out of phase, the last term on the right-hand side integrates to zero and makes no contribution to the mean wave power output. Hence we have

$$\langle \boldsymbol{I} \rangle = -\rho_w\left\langle \frac{\partial\phi}{\partial\tau}\nabla\phi \right\rangle, \tag{3.4}$$

where the symbol $\langle\ \rangle$ stands for the mean value over time.

Equation (3.4) is a familiar formula in acoustics. Evidently the introduction of ϕ by (2.3) implies an acoustic analogy with our stratified density problem; the real velocity potential ψ is analogous to that in a uniform mean density situation with an equivalent potential function ϕ. From (3.3), it is also clear that the response velocity field consists of two parts. The first one, characterized by the analogous velocity $\nabla\phi$, contains all wave motions. The second part is simply the gravitational tendency to induce vertical motions; in our constant-entropy model of the ocean this is a conservative process.

For harmonic excitations, (3.4) can be simplified to

$$\langle I\rangle = \tfrac{1}{2}\omega\rho_{\mathrm{w}}\,\mathrm{Im}\,(\phi^*\nabla\phi), \tag{3.5}$$

where the star implies complex conjugate and $\mathrm{Im}\,(z)$ denotes the imaginary part of z. Now we calculate the wave power radiated by the point source by integrating (3.5), together with the results obtained in the previous section, over a large hemisphere centred at the origin. By letting its radius tend to infinity, we can use the far-field approximation to evaluate the sound power. The far-field sound is, from (2.14),

$$\phi_{\text{sound}} = \frac{-1}{4\pi R}\,\mathrm{e}^{\mathrm{i}\lambda R}[\mathrm{e}^{\mathrm{i}h\lambda\cos\theta}+C_{\mathrm{r}}\,\mathrm{e}^{-\mathrm{i}h\lambda\cos\theta}] \quad \text{as } R\rightarrow\infty, \tag{3.6}$$

where $R = |\boldsymbol{y}|$, $\cos\theta = y_3/R$ and C_{r} is defined by (2.15). On substituting (3.6) into (3.5) and integrating over a hemisphere of radius R, the acoustic power, denoted by W_{a}, is found to be

$$W_{\mathrm{a}} = \frac{\omega\lambda\rho_{\mathrm{w}}}{8\pi}\int_0^1\left[1-\mathrm{Re}\left(\frac{2\omega_0^2-\omega^2+\mathrm{i}g\lambda\xi}{2\omega_0^2-\omega^2-\mathrm{i}g\lambda\xi}\,\mathrm{e}^{2\mathrm{i}h\lambda\xi}\right)\right]\mathrm{d}\xi, \tag{3.7}$$

where $\mathrm{Re}\,(z)$ is the real part of z.

The surface wave power can be similarly calculated from the surface wave field (2.11). A direct substitution into (3.5) and a recognition of the Wronskian (Watson 1966) lead to the simple expression

$$\langle I\rangle_{\mathrm{s}} = \frac{\omega\rho_{\mathrm{w}}}{4\pi r}\left(\frac{\omega^2-2\omega_0^2}{g}\right)^2\exp\left[2(y_3-h)\frac{\omega^2-2\omega_0^2}{g}\right].$$

It follows, by integrating this over a half infinite cylindrical surface whose axis coincides with the negative y_3-axis, that the surface wave power from the point source is

$$W_{\mathrm{s}} = \frac{\omega\rho_{\mathrm{w}}(\omega^2-2\omega_0^2)}{4g}\exp\left(-2h\frac{\omega^2-2\omega_0^2}{g}\right). \tag{3.8}$$

The sound power (3.7) and surface wave power (3.8) can also be obtained by the method suggested by Levine (1980), which starts by multiplying the basic equation (2.4) by $\tfrac{1}{2}\omega\rho_{\mathrm{w}}\phi^*$ and taking the imaginary part of the result, so that

$$\nabla[\tfrac{1}{2}\omega\rho_{\mathrm{w}}\,\mathrm{Im}\,(\phi^*\nabla\phi)] = \tfrac{1}{2}\omega\rho_{\mathrm{w}}\,\delta(y_\alpha, y_3+h)\,\mathrm{Im}\,(\phi^*). \tag{3.9}$$

The term in the bracket can be recognized as (3.5), the wave energy flux. So, integrating (3.9) over the whole water body and applying the divergence theorem, we find the total power delivered to infinity is

$$W_{\text{total}} = \tfrac{1}{2}\omega\rho_{\mathrm{w}}\,\mathrm{Im}\,[\phi^*(0,0,-h)].$$

By using (2.8), it is easy to show that

$$W_{\text{total}} = \frac{\omega\lambda\rho_{\mathrm{w}}}{8\pi}\int_0^1\left[1-\mathrm{Re}\left(\frac{2\omega_0^2-\omega^2+\mathrm{i}g\lambda\xi}{2\omega_0^2-\omega^2-\mathrm{i}g\lambda\xi}\,\mathrm{e}^{2\mathrm{i}h\lambda\xi}\right)\right]\mathrm{d}\xi$$
$$-\frac{\omega\rho_{\mathrm{w}}}{8\pi}\,\mathrm{Im}\left[\int_\lambda^\infty\frac{1}{(k^2-\lambda^2)^{\frac{1}{2}}}\,\frac{2\omega_0^2-\omega^2-g(k^2-\lambda^2)^{\frac{1}{2}}}{2\omega_0^2-\omega^2+g(k^2-\lambda^2)^{\frac{1}{2}}}\exp\left[-2h(k^2-\lambda^2)^{\frac{1}{2}}\right]k\,\mathrm{d}k\right].$$

The first term is exactly the same as (3.7), the sound power. At first glance, the second integral seems purely real, which would suggest zero surface wave power, but this is certainly not the case. The integration path must be indented below the pole at

$k = \omega^2/g$ to comply with the radiation condition. The imaginary part of this integral is 2π times half of the residue at $k = \omega^2/g$, which is exactly (3.8), as is expected.

Now, we return to the calculation of acoustic power. The calculation can be facilitated by expressing (3.7) in the form

$$W_a = \frac{\omega\lambda\rho_w}{8\pi}\left[1+\frac{\sin 2h\lambda}{2h\lambda}-2\frac{\omega^2-2\omega_0^2}{\lambda}\left((\omega^2-2\omega_0^2)\,F(h)-\tfrac{1}{2}g\frac{\partial F(h)}{\partial h}\right)\right] \tag{3.10}$$

where

$$F(h) = \int_0^\lambda \frac{\cos(2h\xi)}{(\omega^2-2\omega_0^2)^2+g^2\xi^2}\,d\xi.$$

In Appendix B, it is found that

$$\left[(\omega^2-2\omega_0^2)\,F(h)-\tfrac{1}{2}g\frac{\partial F(h)}{\partial h}\right] = \frac{\pi}{g}\exp\left(2h\frac{2\omega_0^2-\omega^2}{g}\right)$$
$$-\frac{1}{g}\,\mathrm{Im}\left[E_1\left(2h\frac{2\omega_0^2-\omega^2}{g}+2\mathrm{i}\lambda h\right)\right] \quad \text{when } h \neq 0,$$
$$= \frac{1}{g}\,\mathrm{arctg}\left(\frac{g\lambda}{\omega^2-2\omega_0^2}\right) \quad \text{when } h = 0,$$

with $E_1(z)$ being the exponential integral. Therefore the acoustic power is

$$W_a = \frac{\omega\lambda\rho_w}{8\pi}\left[1+\frac{\sin(2h\lambda)}{2h\lambda}-2\frac{\omega^2-2\omega_0^2}{g\lambda}\exp\left(2h\frac{2\omega_0^2-\omega^2}{g}\right)\right.$$
$$\left.\times\left(\pi+\mathrm{Im}\,E_1\left(2h\frac{2\omega_0^2-\omega^2}{g}+2\mathrm{i}\lambda h\right)\right)\right] \quad \text{when } h \neq 0, \tag{3.11}$$
$$= \frac{\omega\lambda\rho_w}{4\pi}\left(1-\frac{\omega^2-2\omega_0^2}{g\lambda}\,\mathrm{arctg}\frac{g\lambda}{\omega^2-2\omega_0^2}\right) \quad \text{when } h = 0. \tag{3.12}$$

These results are illustrated in figure 2, where the sound power, normalized by $\omega\lambda\rho_w/8\pi$, the power that would be radiated by the source in an infinite space, is plotted as a function of the dimensionless source position $h\omega/c_w$ for some frequencies. Obviously the curves approach the vanishing gravity limit as frequency tends to infinity; high-frequency sound is decoupled from surface waves.

An intriguing influence of gravity on low-frequency sound is also clear in figure 2; the total sound power radiated by the source can be vanishingly small when the source is located at a particular non-zero depth in the water, a striking feature that is never true of the gravity-free problem. This phenomenon can be explained by the change of reflective properties of the sea surface in this uniform gravity model, which provides an additional reflected field that interferes destructively with the original. The non-zero depth h_0, at which the source radiates negligible sound, can be determined by letting $\partial W_a/\partial h = 0$ and $W_a \approx 0$, which leads to the simple equation

$$\frac{\lambda g}{2\omega_0^2-\omega^2}\,\frac{2\lambda h_0\cos(2\lambda h_0)-\sin(2\lambda h_0)}{(2\lambda h_0)^2}+\frac{\sin(2\lambda h_0)}{2\lambda h_0} = 1.$$

This is shown in figure 3. The depth h_0 has been normalized by the acoustic wavelength c_w/ω and plotted versus the dimensionless frequency ω/ω_0. It can be seen that this effect is most important at low frequencies. As ω decreases, h_0 increases until $\omega^2 = 2\omega_0^2$. Below this value h_0 does not exist because the ocean surface in that case becomes acoustically rigid; the image field is then essentially constructive.

The ratio of (3.11) to (3.8), that is, the ratio of acoustic power to surface-wave power,

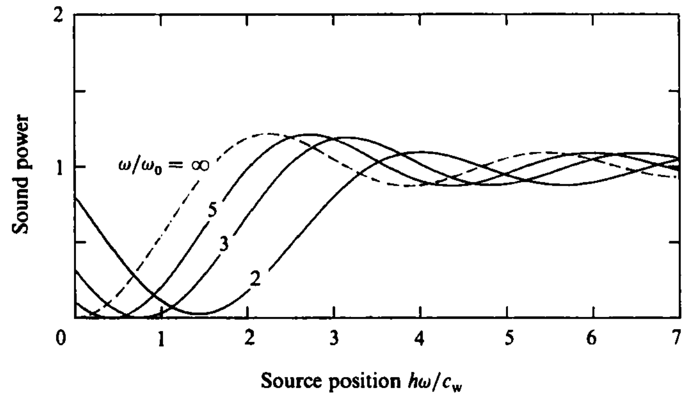

FIGURE 2. Surface amplification of sound power from a harmonic point source. The sound power is normalized by the power of the same source in an infinite space.

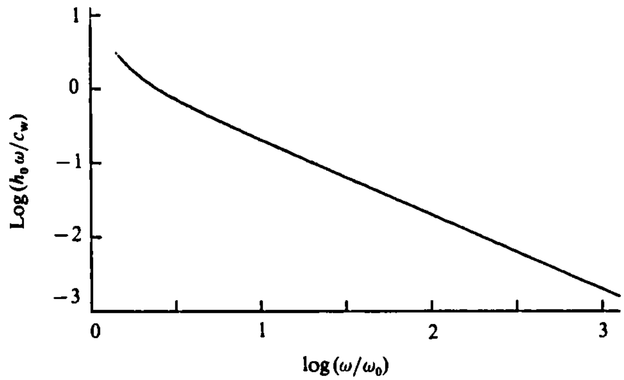

FIGURE 3. The particular depth at which the source can radiate only a negligible sound.

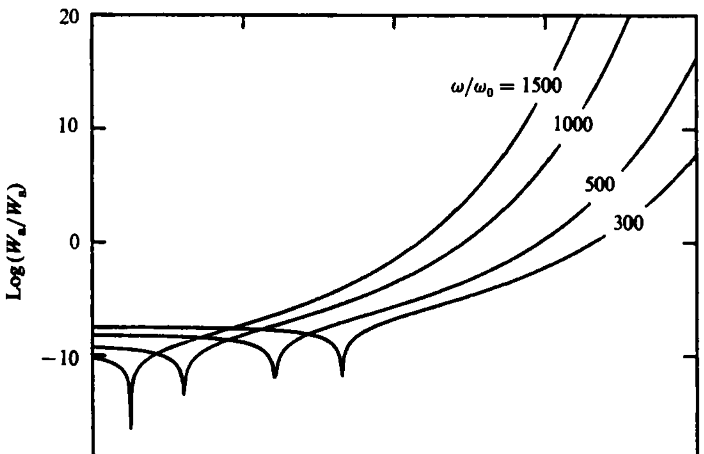

FIGURE 4. The power ratio of sound to surface waves from a harmonic point source. The downwards spikes indicate that the source is located at the depth h_0.

is shown in figure 4 as a function of the reduced source position for some values of ω/ω_0. Apparently, deep water sources are much more efficient in sound radiation than in surface-wave production, and more energy is assigned to surface waves as the source approaches the surface. The downwards spikes of the curves correspond to the situations where the source is located at h_0. The acoustic power in those cases is negligibly small.

4. Waves generated by surface pressure fluctuations

The most important sources of surface waves are winds and atmospheric pressure fluctuations moving across the ocean surface (Phillips 1977). Isakvich & Kur'yanov (1970) have interpreted empirical evidence of surface waves to suggest that the wind-induced pressure fluctuations may also be the cause of the underwater noise within the frequency range 10–50 Hz. Here we demonstrate through a definite theoretical modelling that this mechanism can also generate a considerable low-frequency sound below 10 Hz. At low frequencies, the surface wavelength is usually much bigger than the wave height and the slope of the ocean-surface elevation is usually very small. Hence the surface pressure fluctuations can be effectively regarded as an external distribution of force acting on the mean position of the ocean surface, that is, $y_3 = 0$. In this view, the boundary condition at $y_3 = 0$ becomes, if we let $Q(y_\alpha, \tau)$ be the surface pressure distribution,

$$\frac{\partial^2 \psi}{\partial \tau^2} + g\frac{\partial \psi}{\partial y_3} = -\frac{1}{\rho_w}\frac{\partial Q}{\partial \tau},$$

which, together with the governing equation (3.1) and the radiation condition at $y_3 \to -\infty$, can be used to determine ψ. By making use of the transformation (2.3) and following the procedure of §2, the far-field sound can be derived as

$$\phi_a = \frac{-x_3}{(2\pi)^2 c_w \rho_w |\boldsymbol{x}|^2} \int_\infty Q(y_\alpha, \tau) \exp\left(\mathrm{i}\frac{\omega}{c_w}\left[|\boldsymbol{x}| - \frac{x_\alpha y_\alpha}{|\boldsymbol{x}|}\right]\right) \mathrm{e}^{\mathrm{i}\omega(t-\tau)}\, \mathrm{d}^2 y_\alpha\, \mathrm{d}\tau\, \mathrm{d}\omega,$$

from which the acoustic energy flux in the radial direction can be derived as

$$I_a = \frac{\omega^2 x_3^2}{2(2\pi)^3 c_w^3 \rho_w |\boldsymbol{x}|^4} \int_\infty G(y'_\alpha, -\omega) \exp(\mathrm{i}\xi_\alpha y'_\alpha)\, \mathrm{d}^2 y_\alpha\, \mathrm{d}^2 y'_\alpha, \tag{4.1}$$

where we have denoted $\omega x_\alpha / c_w |\boldsymbol{x}|$ by ξ_α and the ω-integral has been dropped so that I_a is the value of the energy flux in a unit frequency band. The function $G(y'_\alpha, \omega)$, y'_α being the space separation, is defined as the Fourier transform with respect to the time delay of the cross-correlation of the source function $Q(y_\alpha, \tau)$.

Pressure fluctuations on the ocean surface have been a research subject in ocean-wave studies for a long time due to their importance in producing surface waves. Though the detailed structure of their distribution is not known and a precise description of them is not possible, a reasonably good representation can be extrapolated from measurements. Snyder *et al.* (1981) have suggested that

$$G(y'_\alpha, \omega) = A(\omega) \exp[-((\kappa_1 y'_1)^2 + (\kappa_2 y'_2)^2)^{\frac{1}{2}}] \exp(-\mathrm{i}k_0 y'_1), \tag{4.2}$$

where κ_1 and κ_2 are, respectively, the downwind and crosswind coherence and k_0 is the downwind wavenumber. We suppose that this source extends over a linear dimension L that is much bigger than the coherence scales, so that the source can be regarded as being homogeneously distributed within $|y_\alpha| < L$ and vanishing outside it. Hence, substituting (4.2) into (4.1) and carrying out the y_α-integration in the region $|y_\alpha| < L$, we find

$$I_a = \frac{\omega^2 x_3^2 L^2 A(\omega)}{4(2\pi)^2 c_w^3 \rho_w |x|^4} \int_\infty \exp\left[-((\kappa_1 y_1')^2 + (\kappa_2 y_2')^2)^{\frac{1}{2}}\right] \exp\left[\mathrm{i}(\xi_\alpha y_\alpha' + k_0 y_1')\right] \mathrm{d}y_\alpha'.$$

The y_α'-integral can be carried out explicitly to give

$$\frac{2\pi}{\kappa_1 \kappa_2}\left[1 + \left(\frac{k_0 + \xi_1}{\kappa_1}\right)^2 + \left(\frac{\xi_2}{\kappa_2}\right)^2\right]^{-\frac{3}{2}}. \tag{4.3}$$

Since ξ_α/κ_α is basically of the same order as the wind Mach number $U/c_w \ll 1$, the square bracket in this is effectively $1 + k_0^2/\kappa_1^2$. The sound power can then be evaluated by integrating I_a over a hemisphere centred at the origin, which yields

$$W_a = \frac{\omega^2 L^2 A(\omega)}{12\pi c_w^3 \rho_w \kappa_1 \kappa_2}\left[1 + \frac{k_0^2}{\kappa_1^2}\right]^{-\frac{3}{2}}. \tag{4.4}$$

The surface wave power can be derived in the same way from the surface potential

$$\phi_s = \frac{1}{2\pi g^2 \rho_w}\left(\frac{1}{2\pi g |x_\alpha|}\right)^{\frac{1}{2}} \int_\infty \exp\left[\frac{\omega^2}{g} x_3 + \mathrm{i}\left(\frac{\omega^2}{g}\left(|x_\alpha| - \frac{x_\alpha y_\alpha}{|x_\alpha|}\right) - \tfrac{1}{4}\pi\right)\right]$$
$$\times\, \omega^2 Q(y_\alpha, \tau)\, \mathrm{e}^{\mathrm{i}\omega(t-\tau)}\, \mathrm{d}^2 y_\alpha\, \mathrm{d}\tau\, \mathrm{d}\omega,$$

which yields the surface wave energy flux in unit frequency band as

$$I_s = \frac{\omega^7}{2(2\pi)^2 g^4 \rho_w |x_\alpha|} \exp\left(2\frac{\omega^2}{g} x_3\right) \int_\infty G(y_\alpha', \omega) \exp\left(\mathrm{i}\frac{\omega^2}{g}\frac{x_\alpha y_\alpha'}{|x_\alpha|}\right) \mathrm{d}^2 y_\alpha\, \mathrm{d}^2 y_\alpha'.$$

Obviously the double integration in this result is identical to that in (4.1) provided ξ_α is now replaced by $\omega^2 x_\alpha / g|x_\alpha|$. Hence we can make use of the result (4.3) to evaluate I_s. By integrating the result over a half infinite cylindrical surface of radius L and neglecting terms smaller by a factor $g/\omega U$, we find the surface wave power as

$$W_s = \frac{L^2 A(\omega) \kappa_1^2}{8\omega \rho_w \kappa_2} \int_0^{2\pi} \left[\cos^2\theta + \frac{\kappa_1^2}{\kappa_2^2}\sin^2\theta\right]^{-\frac{3}{2}} \mathrm{d}\theta,$$

which, together with (4.4), yields the power ratio

$$\frac{W_a}{W_s} = \frac{2\omega^3}{3\kappa_1^3 c_w^3 \mu [1 + k_0^2/\kappa_1^2]^{\frac{3}{2}}}, \tag{4.5}$$

where $$\mu = \int_0^{2\pi} \left(\cos^2\theta + \frac{\kappa_1^2}{\kappa_2^2}\sin^2\theta\right)^{-\frac{3}{2}} \mathrm{d}\theta.$$

This result reveals the linear relation between sound and surface wave power output from the same source $Q(y_\alpha, \tau)$. The power ratio is determined by the coherences of their common source. If we assume that k_0, κ_1 and κ_2 are all of the same order, μ simply reduces to 2π and $1 + k_0^2/\kappa_1^2 = 2$, so that the right-hand side of (4.5) can be seen to be essentially independent of frequency if we further assume that the downwind coherence κ_1 is proportional to ω/U. The sound power in this case increases simply in proportion to the surface wave power. If the surface wave activity increases due to the increase of the source strength, the underwater sound also increases by the same amount. The acoustic field is not dependent on surface waves in any nonlinear way, as in the view that considers surface waves as the source of the sound.

In what follows, we justify this conclusion by comparing the calculated power ratio (4.5) with that measured in the natural ocean.

Sound in the ocean has been measured on many occasions, but the measurements are usually of pressure spectra. Since we choose to work with the wave power instead of pressure spectra, it is necessary to first deduce the acoustic power output from those measured pressure spectra. In the ocean, measurements are usually made in the water right below a wind field of large dimension. We choose a cylindrical surface of radius L which just encloses the source region so that the total acoustic power is given by the sum of the power adsorbed by the bottom and that crossing the cylindrical surface.

There is experimental evidence that sea-bottom reflection is significant at low frequencies; the pressure reflection coefficient, which we denote by β, is close to unity (Urick 1967). Letting $P(\omega)$ be the total pressure spectrum, the spectrum of the incident pressure can then be written as $P(\omega)/(1+\beta)^2$, which is actually the incident wave energy flux if it is divided by $c_w \rho_w$. Since the energy loss through unit area of the sea floor is $1-\beta^2$ times the incident energy flux, the total power loss on the bottom, which is denoted by W_1, is

$$W_1 = \int_s \frac{P(\omega)}{c_w \rho_w} \frac{1-\beta}{1+\beta} \mathrm{d}s,$$

where s is the sea bottom surface of radius L. Since this area is right below the source region, it is reasonable to assume that $P(\omega)$ is uniform over it, provided L is relatively large in comparison to the acoustic wavelength, which is the case in the natural ocean. In general, β is a function of both frequency and the angle of incidence, but here we simply regard it as the mean value over all possible angles. In this case, W_1 reduces to

$$W_1 = \frac{P(\omega)}{c_w \rho_w} \frac{1-\beta}{1+\beta} \pi L^2. \tag{4.6}$$

The sound power crossing the chosen cylindrical surface can be evaluated by integrating the energy flux, which depends only on the vertical coordinate because of the symmetrical geometry. At frequency ω the value of the energy flux is given by the pressure spectrum divided by $c_w \rho_w$ and multiplied by a directional factor which accounts for the fact that energy rays received on the cylindrical surface are not necessarily perpendicular to it. We will not consider the complicated details of this factor, but simply take it as unity. This approximation is justifiable because the source region is quite extensive, so that the angle between the impinging energy ray and the normal to the control surface is very small for most of the main contributing energy rays that come from the direct and the first reflected radiation from distant sources. Hence the sound power crossing the cylindrical surface, which we denote by W_2, is approximately

$$W_2 = 2\pi L \int_{-H}^{0} \frac{P(\omega)}{c_w \rho_w} f(y_3)\, \mathrm{d}y_3, \tag{4.7}$$

where H is the depth of the ocean. In this we have taken account of the variation of sound pressure with ocean depth by expressing the pressure spectrum at the depth $(-y_3)$ by $P(\omega)f(y_3)$, the spectrum on the bottom modified by a factor $f(y_3)$ that can be derived from experiments. In the low-frequency region we find from Morris' (1978) measurements

$$f(y_3) = \exp[3.07 \times (y_3 + H)] \quad (-H < y_3 < -\lambda_s),$$

where λ_s is the surface wavelength and y_3 and H are measured in metres. The function

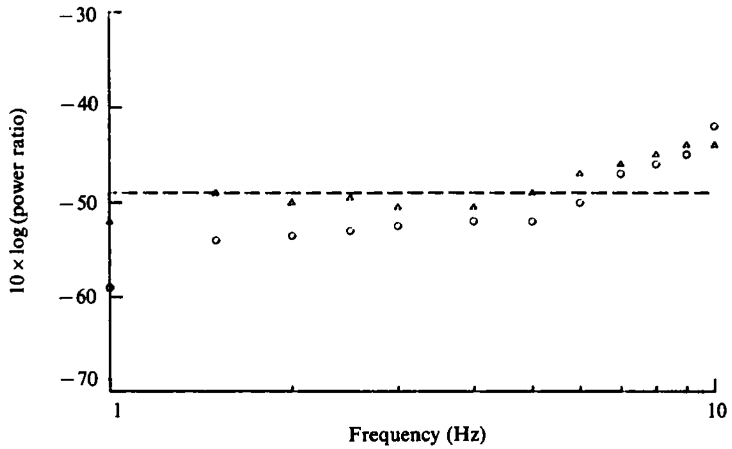

FIGURE 5. The power ratio of sound to surface waves driven by surface pressure fluctuations on the ocean surface. The theory (4.5) is indicated by the dashed line, and the two sets of experimental data were measured respectively 300 m (circles) and 1200 m (triangles) below the ocean surface.

$f(y_3)$ vanishes in the region $y_3 > -\lambda_s$ to comply with the fact that no sound wave can propagate along a pressure-release surface. On substituting this into (4.7) we find that

$$W_2 = 2\pi LH \frac{P(\omega)}{c_w \rho_w} B(H), \tag{4.8}$$

where $$B(H) = \frac{3.26 \times 10^3}{H} [\exp(3.07 \times 10^{-4} H) - 1].$$

Now the total acoustic power output W'_a can be expressed in terms of $P(\omega)$ by the addition of (4.6) and (4.8):

$$W'_a = \frac{P(\omega)}{c_w \rho_w} 2\pi LH \left[B(H) + \frac{L}{2H} \frac{1-\beta}{1+\beta} \right]. \tag{4.9}$$

The surface wave power W'_s in the ocean can be calculated directly from the surface wave spectrum $\Phi(\omega)$, which has been measured and modelled extensively in ocean wave studies. Surface wave energy has a density of the form $\rho_w g \Phi(\omega)$ per unit length per unit frequency band, and propagates at the group velocity $g/2\omega$. The product of the two gives the energy flux, and the surface wave power can be derived as

$$W'_s = \frac{\rho_w g^2}{2\omega} \Phi(\omega) 2\pi L,$$

which, together with (4.9), gives the sound to surface-wave power ratio in the ocean in terms of sound pressure spectrum and surface wave spectrum as

$$\frac{W'_a}{W'_s} = \frac{2\omega H}{c_w \rho_w^2 g^2} \frac{P(\omega)}{\Phi(\omega)} \left[B(H) + \frac{L}{2H} \frac{1-\beta}{1+\beta} \right]. \tag{4.10}$$

Now the theoretically calculated power ratio (4.5) can be compared with this result, which is illustrated in figure 5. The measured ratio is plotted according to (4.10) by making use of the measurements of $P(\omega)$ from Nichols (1981) and a surface wave spectrum derived by Phillips (1985), namely, $\Phi(\omega) = \alpha u_* g/\omega^4$, where α is a dimensionless constant determined by experimental data, equal to 0.01×6–0.01×11, and u_* is the friction velocity that is approximately related to the wind speed U, measured

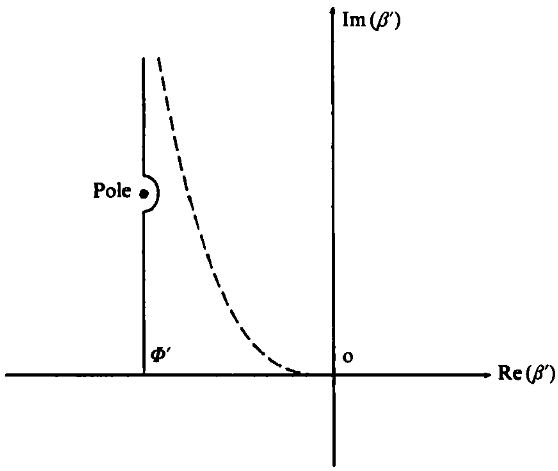

FIGURE 6. The deformation of the integration path Γ' onto the steepest descent path (A 3).

at 10 metres above the water surface, by $U = 11.0 \times u_*\{2.0-\log u_*\}$ (Phillips 1977). In plotting (4.10), the second term in the square bracket has been neglected (because $1-\beta$ is very small) as being much smaller than the first term for comparable L and H. We take the uniform wind speed U as 20 knots, corresponding to the sea state when $P(\omega)$ was measured. The Phillips spectrum in this case is valid in the frequency range of about 0.1–13 Hz, within which our comparison is made. The downwind coherence κ_1 has been taken as being proportional to ω/U with a proportionality constant of 0.1 according to Snyder *et al.* (1981), their estimate of which is 0.1 to 0.4. Evidently, the theory, which is represented by the dashed line in figure 5, is in good agreement with the measured data in both structure and order of magnitude.

5. Conclusions

Sound and surface waves radiated from a defined source have been studied in detail. In this particular model, which takes account of both gravitational effects and fluid compressibility, sound and surface waves co-exist. Surface waves are unmodified by fluid compressibility; their generation and propagation show the same character as in incompressible flow. Sound radiation, however, is greatly augmented, particularly at low frequencies, by gravity waves which change the reflection coefficient of the sea surface from -1 in gravity-free situations to a function of both frequency and observation angle. This reflection coefficient limits to -1 as frequency tends to infinity, but in the low-frequency region it differs from -1 considerably.

Energy calculations show that the source induces, apart from the ordinary wave motions, a background oscillation which has no influence on wave power output. The wave power from the point source is evaluated explicitly. The way in which the source assigns its energy between sound and surface waves shows that deep-water sources are more efficient in sound generation than in surface-wave production, while sources adjacent to the surface radiate more energy to surface waves than to sound. An unusual phenomenon has been found in this model; a submerged source may radiate a negligible sound at a particular non-zero depth. This non-zero depth has been found to have significant values, again, in the low-frequency region.

Particular attention has been paid to the situation where the sources are pressure fluctuations moving across the ocean surface. The calculated energy ratio of sound to surface waves has been compared with that from measurements, which reveals good consistency. It is then concluded that the direct radiations from sources that induced surface waves may contribute an appreciable amount to the observed low-frequency ocean noise.

The author would like to thank his supervisor, Professor J. E. Ffowcs Williams for suggesting the problem and for his encouragement. The financial support from the University of Cambridge (Chancellor's Fund) is also acknowledged.

Appendix A

In this Appendix, we calculate the integral ϕ_a specified by (2.10). For convenience, we transfer the Hankel function into a Bessel function and further express the Bessel function as an integral. This leads to

$$\phi_a = \frac{\omega^2 - 2\omega_0^2}{2i\pi^2} \int_0^\pi \int_0^\infty \frac{1}{\gamma(2\omega_0^2 - \omega^2 - ig\gamma)} e^{-i[\gamma(y_3 - h) + kr\cos\alpha]} k \, dk \, d\alpha. \tag{A 1}$$

By a change of variables $k = \lambda \sin\beta$ and by the use of the spherical coordinate system R_2 and θ_2, (A 1) can be rewritten as

$$\phi_a = \frac{i\lambda(2\omega_0^2 - \omega^2)}{2\pi^2} \int_\Gamma \int_0^\pi e^{-i\lambda R_2[\cos\beta\cos\theta_2 + \sin\beta\sin\theta_2\cos\alpha]} \frac{\sin\beta}{2\omega_0^2 - \omega^2 - ig\lambda\cos\beta} d\beta \, d\alpha. \tag{A 2}$$

The integration path Γ in the complex β-plane is $(0 + i0) \rightarrow (\frac{1}{2}\pi + i0) \rightarrow (\frac{1}{2}\pi - i\infty)$, which is determined by both the change of variables, $k = \lambda \sin\beta$, and the specification of γ. Now, introduce a symmetrical transformation

$$\begin{cases} \cos\beta' = -\cos\beta\cos\theta_2 - \sin\beta\sin\theta_2\cos\alpha \\ \cos\beta = -\cos\beta'\cos\theta_2 - \sin\beta'\sin\theta_2\cos\alpha' \end{cases}$$

to convert the integral (A 2) from the (α, β)-space to (α', β')-space. For such a symmetrical transformation, it can be shown that the Jacobian is -1. Therefore

$$\phi_a = \frac{i\lambda(\omega^2 - \omega_0^2)}{2\pi^2} \int_{\Gamma'} \int_0^\pi \frac{\exp\{i\lambda R_2\cos\beta'\}\sin\beta'}{2\omega_0^2 - \omega^2 + ig\lambda\cos\beta'\cos\theta_2 + ig\lambda\sin\beta'\sin\theta_2\cos\alpha'} d\alpha' \, d\beta'$$

where the integration path Γ' in the β'-plane correspondingly becomes $(0 + i0) \rightarrow (-\frac{1}{2}\pi + i0) \rightarrow (-\frac{1}{2}\pi + i\infty)$. The integration with respect to α' can be carried out explicitly to give

$$\phi_a = \frac{\lambda(\omega^2 - 2\omega_0^2)}{2\pi i} \int_{\Gamma'} \frac{\sin\beta'\exp\{i\lambda R_2\cos\beta'\}}{\{[2\omega_0^2 - w^2 + ig\lambda\cos\beta'\cos\theta_2]^2 + g^2\lambda^2\sin^2\beta'\sin^2\theta_2\}^{\frac{1}{2}}} d\beta'.$$

Now, we deform the path Γ' onto the steepest descent path that can be expressed by the parametric equation

$$\cos\beta' = 1 + is, \tag{A 3}$$

where s is a real variable varying from zero to infinity as Γ' goes from zero to $-\frac{1}{2}\pi + i\infty$. This path is schematically shown in figure 6.

It should be mentioned that, when deforming the path Γ' onto (A 3), one needs to check if any poles are crossed. This problem does not arise in our calculations since we have already suitably dealt with the poles in deriving ϕ_a in §2, so that all poles are outside the area enclosed by Γ' and (A 3). Therefore, we eventually have

$$\phi_a = \frac{\lambda(\omega_0^2-\omega^2)}{2\pi}\,e^{i\lambda R_2}\int_0^\infty E(s)\,e^{-\lambda R_2 s}\,ds,$$

with $E(s)$ given by (2.13). The result (2.12) then follows directly from this by expanding $E(s)$ at $s = 0$ and integrating the result term by term; the result is not a convergent series, but is asymptotic as $R_2 \to \infty$, with an error smaller than any inverse power of R_2.

Appendix B

From (3.10), we have

$$F(h) = \int_0^\lambda \frac{\cos(2h\xi)}{(\omega^2-2\omega_0^2)^2+g^2\xi^2}\,d\xi, \tag{B 1}$$

whose first- and second-order derivatives with respect to h are, respectively,

$$\frac{\partial F(h)}{\partial h} = \int_0^\lambda \frac{-2\xi\sin(2h\xi)}{(\omega^2-2\omega_0^2)^2+g^2\xi^2}\,d\xi, \tag{B 2}$$

and

$$\frac{\partial^2 F(h)}{\partial h^2} = \frac{2\sin(2h\lambda)}{-hg^2}+\frac{4(\omega^2-2\omega_0^2)^2}{g^2}\int_0^\lambda \frac{\cos(2h\xi)}{(\omega^2-2\omega_0^2)^2+g^2\xi^2}\,d\xi. \tag{B 3}$$

Equations (B 1) and (B 3) can be combined to form a second-order ordinary differential equation, with the boundary conditions at $h = 0$ determined by (B 1) and (B 2), namely,

$$\frac{\partial^2 F(h)}{\partial h^2}-\frac{4(\omega^2-2\omega_0^2)^2}{g^2}F(h) = \frac{2\sin(2h\lambda)}{-hg^2},$$

$$F(0) = \frac{1}{g(\omega^2-2\omega_0^2)}\,\mathrm{arctg}\,\frac{g\lambda}{\omega^2-2\omega_0^2},$$

and

$$\frac{\partial F}{\partial h}(0) = 0.$$

It is a straightforward way to solve this problem by the method of Green functions. The result is

$$F(h) = \frac{1}{2g(\omega^2-2\omega_0^2)}\left[\mathrm{arctg}\,\frac{g\lambda}{\omega^2-2\omega_0^2}\exp\left(2h\frac{2\omega_0^2-\omega^2}{g}\right) + \int_0^\infty \exp\left(2|h-h'|\frac{2\omega_0^2-\omega^2}{g}\right)\frac{\sin(2\lambda h')}{h'}\,dh'\right].$$

Therefore we have

$$\left[(\omega^2-2\omega_0^2)\,F(h)-\tfrac{1}{2}g\frac{\partial F(h)}{\partial h}\right] = \frac{1}{g}\,\mathrm{arctg}\,\frac{g\lambda}{\omega^2-2\omega_0^2}\exp\left(2h\frac{2\omega_0^2-\omega^2}{g}\right) + \frac{1}{g}\int_0^h \frac{\sin(2\lambda h')}{h'}\exp\left[2(h-h')\frac{2\omega_0^2-\omega^2}{g}\right]dh'.$$

When $h = 0$ the h'-integral is zero and when $h \neq 0$ we have

$$\int_0^h \frac{\sin(2\lambda h')}{h'} \exp\left(2h' \frac{2\omega_0^2 - \omega^2}{g}\right) \mathrm{d}h' = \pi - \operatorname{arctg} \frac{\lambda g}{\omega^2 - 2\omega_0^2} + \operatorname{Im}\left[E_1\left(-2\frac{\omega^2 - 2\omega_0^2}{g} + 2\mathrm{i}\lambda h\right)\right],$$

where $E_1(z)$ is the exponential integral. This immediately gives the result quoted in §3.

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